

FACTORIZATION OF DIAGONALLY DOMINANT OPERATORS ON $L_1([0, 1], X)$

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ABSTRACT. Let X be a separable Banach space. It is shown that every diagonally dominant invertible operator on $L_1([0, 1], X)$ can be factored uniquely as a product of an invertible upper triangular operator and an invertible unit lower triangular operator.

Recently, the problem of factorization of operators on Banach spaces has attracted a great deal of attention from various workers in analysis. Operator algebraists, for example, have considered the problem of factoring an operator (usually positive definite) into a product of the form T^*T , where T belongs to a given nest algebra [1, 6]. Approximation theorists, on the other hand, who view this problem as one of "infinite dimensional numerical analysis" have obtained factorization theorems for totally positive operators and Toeplitz totally positive matrices [3, 4]. These theorems have proven useful in connection with spline interpolation problems [5] and certain time invariant linear systems [8].

Factorization of matrices arises naturally in the solution of systems of linear equations via Gauss elimination. In fact, if an invertible matrix can be factored into a product of an upper triangular and a lower triangular matrix (with respect to a fixed basis), then the upper triangular matrix can be obtained by Gauss elimination. Moreover, it is easily checked that an invertible $n \times n$ matrix has such a factorization if and only if the compressions of the matrix to the span of the first k basis elements ($k = 1, 2, \dots, n$) are invertible. Among the matrices which satisfy this compression criterion are positive definite matrices viewed as operators on l_2^n and invertible diagonally dominant matrices viewed as operators on l_1^n .

Motivated by this last case, the second author and P. W. Smith were able to show that any bounded (column) diagonally dominant operator T on l_1 can be factored uniquely as a product $T = LU$, where the operators L, L^{-1} are unit lower triangular and the operators U, U^{-1} are upper triangular with respect to the canonical basis of l_1 [12]. Such a factorization is called an LU -factorization. The object of this paper is to prove an analogous result for operators on the space $L_1([0, 1], X)$ of Lebesgue-Bochner integrable functions on $[0, 1]$ with values in a separable Banach space X . We first show that there are natural classes of operators on $L_1([0, 1], X)$ to which the terms diagonally dominant, upper triangular and lower triangular may and will be applied. We then show that a diagonally dominant invertible operator on $L_1([0, 1], X)$ has a unique LU -factorization. We base our work on a

Received by the editors May 15, 1984 and, in revised form, December 19, 1984. Presented to the Society on January 9, 1985.

1980 *Mathematics Subject Classification*. Primary 47A68; Secondary 46E40.

Key words and phrases. Diagonally dominant, triangular, invertible.

representation theorem of Kalton [10] for operators on $L_1([0, 1], X)$. Our result is somewhat surprising in view of the fact that continuous analogues of discrete factorization results do not always hold. For example, Arveson [1] has shown that every positive invertible operator T on l_2 can be factored in the form $T = S^*S$, where S is contained in algebra determined by a discrete linearly ordered nest of projections. But Larson [11] has shown that positive invertible operators on $L_2[0, 1]$ need not have such a factorization relative to the algebra determined by a continuous nest of projections.

Finally, we remark that Barkar and Gohberg [2] have done extensive work on the factorization question and have given an abstract set of necessary and sufficient conditions for an operator to factor. Unfortunately, we find it difficult to apply these criteria in the present case.

The rest of this paper is divided into three sections. The first section gives some basic definitions and fixes some notation; the second collects the necessary facts on diagonal and triangular operators that are required in §III which contains the proof of the factorization result.

I. Definitions and notation. Throughout this paper, the letter X will denote a separable Banach space. If Y and Z are Banach spaces, we will denote the space of all bounded linear operators from Y to Z by $B(Y, Z)$. In the case when $Y = Z$ we will write $B(Y)$ for $B(Y, Y)$. We denote the X -valued Borel measures of finite variation on $[0, 1]$ by $M([0, 1], X)$. If λ is such a measure, then its variation is denoted by $|\lambda|$. The symbol $L_1([0, 1], X)$ represents the Banach space of Lebesgue-Bochner measurable functions on $[0, 1]$ taking values in X . Lebesgue measure on $[0, 1]$ will be denoted by μ . Following Kalton [10], an operator T on $L_1([0, 1], X)$ is said to be *diagonal* if for every Borel set B in $[0, 1]$ and function f in $L_1([0, 1], X)$, we have that $f(\omega) = 0$ for all ω in B implies that $(Tf)(\omega) = 0$ for all ω in B . A map $g: [0, 1] \rightarrow B(Y, Z)$ is said to be strongly Borel measurable if it is a Borel map for the strong operator topology on $B(Y, Z)$. Kalton [10] has shown that for every diagonal operator T on $L_1([0, 1], X)$, there exists a strongly Borel measurable map $g: [0, 1] \rightarrow B(X)$ such that $Tf(\cdot) = g(\cdot)f(\cdot)$ μ -a.e. for all f in $L_1([0, 1], X)$. He also showed that there is a norm one projection $P_d: B(L_1[0, 1], X) \rightarrow B(L_1([0, 1], X))$ onto the subspace of diagonal operators. Hence an operator T in $B(L_1[0, 1], X)$ is said to be *diagonally dominant* if for each Borel subset E of $[0, 1]$ and each x in X we have that $\|P_d(T(x\chi_E))\| \geq \|(T - P_d(T))(x\chi_E)\|$. We note that this is equivalent to requiring that $\|P_d(T)(f)\| \geq \|(T - P_d(T))(f)\|$ for all f in $L_1([0, 1], X)$.

For each Borel subset A of $[0, 1]$, we define a projection operator $P_A: L_1([0, 1], X) \rightarrow L_1([0, 1], X)$ by $Pf = f\chi_A$ for all f in $L_1([0, 1], X)$. (Here χ_A denotes the characteristic function of the set A .) For any projection P on $L_1([0, 1], X)$, let $P^\perp = I - P$ denote its complement. Then an operator T in $B(L_1[0, 1], X)$ is said to be *upper triangular* if $P_{[0, b]}^\perp TP_{[0, b]} = 0$ for all b in $[0, 1]$. Similarly, an operator T is said to be *lower triangular* if $P_{[0, b]} TP_{[0, b]}^\perp = 0$ for all b in $[0, 1]$. An operator T is said to be *strictly upper (lower) triangular* if T is upper (lower) triangular and $P_d(T) = 0$. An operator T is said to be *unit upper (lower) triangular* if it is upper (lower) triangular and $P_d(T) = I$. We say that an invertible operator T in $B(L_1([0, 1], X))$ has an *LU-factorization* if there exist invertible operators L and U such that $T = LU$, and the operators L, L^{-1} are unit lower triangular while the operators U, U^{-1} are upper triangular.

Although the results of this paper are stated in terms of $L_1([0, 1], X)$, where X is a separable Banach space, $X = \mathbb{R}$ is the main case and the reader loses little by thinking only of this case.

II. Diagonal and triangular operators. In this section we collect various facts about projections onto spaces of diagonal and triangular operators which are needed to prove the factorization theorem. We base our work on a representation theorem of Kalton [10], which we include here for easy reference.

THEOREM 1 (KALTON). *Let $T: L_1([0, 1], X) \rightarrow L_1([0, 1], X)$ be a bounded linear operator. Then for each x in X and t in $[0, 1]$ there exists a measure $\mu_{x,t}$ in $M([0, 1], X)$ such that for all f in L_1*

$$(*) \quad T(fx)(t) = \int_{[0,1]} f(s) d\mu_{x,t}(s) \quad \mu\text{-a.e.}$$

Moreover,

(i) *for each x , the map $t \rightarrow \mu_{x,t}$ of $[0, 1]$ into $M([0, 1], X) \subset B(C[0, 1], X)$ is strongly Borel measurable,*

(ii) *the map $x \rightarrow \int_{[0,1]} \cdot d\mu_{x,\cdot}$ is an element of $B(X, B(L_1, L_1([0, 1], X)))$,*

(iii) $\|T\| = \sup_{\|x\| \leq 1} \sup_{\mu_B > 0} (\mu_B)^{-1} \int_{[0,1]} |\mu_{x,t}|(B) d\mu(t).$

Conversely, if for each x in X and t in $[0, 1]$ there exists a measure $\mu_{x,t}$ in $M([0, 1], X)$ so that (i) and (ii) are satisfied, then there exists a unique bounded operator $T: L_1([0, 1], X) \rightarrow L_1([0, 1], X)$ satisfying () and whose norm is given by (iii).*

The collection of measures $\{\mu_{x,t}\}_{x \in X, t \in [0,1]}$ will be called the representing family of measures for T . We note that for each x the measures $\{\mu_{x,t}\}_{x \in X, t \in [0,1]}$ are unique up to null sets.

PROPOSITION 2. (i) *There exists a projection*

$$P_d: B(L_1([0, 1], X)) \rightarrow B(L_1([0, 1], X))$$

of norm 1 onto the subspace of diagonal operators.

(ii) *If $T, D \in B(L_1([0, 1], X))$ and D is a diagonal operator, then $P_d(TD) = P_d(T)D$.*

(iii) *There exist a pair of complementary projections $P_+, P_-: B(L_1([0, 1], X)) \rightarrow B(L_1([0, 1], X))$ of norm 1 onto the subspaces of upper triangular and strictly lower triangular operators, respectively.*

(iv) *If $T \in B(L_1([0, 1], X))$ is a diagonally dominant invertible operator, then $P_d(T)$ is an invertible operator and the inverse is also a diagonal operator.*

PROOF. (i) This is merely a restatement of part of Proposition 6.1 of [10].

(ii) For each Borel subset A of $[0, 1]$, define a projection operator $P_A: L_1([0, 1], X) \rightarrow L_1([0, 1], X)$ by $Pf = f\chi_A$. Now since D is diagonal, there is a strongly Borel measurable function $g: [0, 1] \rightarrow B(X)$ such that for all f_1 in $L([0, 1], X)$, we have that $(Df)(t) = g(t)f(t)$ μ -a.e. and $\|D\| = \|g\|_\infty$ [10, p. 312]. Hence it is easy to see that $P_AD = DP_A$ for each Borel set A . Now suppose, for each n , we partition $[0, 1]$ into disjoint Borel sets $\{B_{n1}, \dots, B_{n,l(n)}\}$ of diameter at most $1/n$ and define operators $\Pi_n: B(L_1([0, 1], X)) \rightarrow B(L_1([0, 1], X))$ by $\Pi_n(T) = \sum_{k=1}^{l(n)} P_{B_{n,k}} T P_{B_{n,k}}$

for $T \in B(L_1([0, 1], X))$. Then $\Pi_n(TD) = \Pi_n(T)D$ and since $\lim_n \Pi_n(T)(f) = P_d(T)(f)$ [10, p. 318] for each f in $L([0, 1], X)$, we have that $P_d(TD) = P_d(T)D$.

(iii) Let T be in $B(L_1([0, 1], X))$ and let $\{\mu_{x,t}\}_{x \in X, t \in [0, 1]}$ be the representing family of measures for T . Define, for each x in X and t in $[0, 1]$, a measure $\mu_{x,t}^+$ in $M([0, 1], X)$ by $\mu_{x,t}^+(E) = \mu_{x,t}(E \cap [t, 1])$. Then, for each x , the map $t \rightarrow \mu_{x,t}^+$ of $[0, 1]$ into $M([0, 1], X)$ is strongly Borel measurable (since, for example, it is the pointwise limit of the strongly Borel measurable maps

$$t \rightarrow \sum_{j=0}^{2^n} \mu_{x,t} \left(\cdot \cap \left[\frac{j}{2^n}, 1 \right] \right) \chi_{[j/2^n, (j+1)/2^n)}(t).$$

Furthermore, if a, b are scalars, x_1, x_2 are elements in X and f is in L_1 , then

$$\begin{aligned} \int_{[0,1]} f(s) d\mu_{ax_1+bx_2,t}^+(s) &= \int_{[t,1]} f(s) d\mu_{ax_1+bx_2,t}(s) \\ &= a \int_{[t,1]} f(s) d\mu_{x_1,t}(s) + b \int_{[t,1]} f(s) d\mu_{x_2,t}(s) \quad \mu\text{-a.e.} \\ &= a \int_{[0,1]} f(s) d\mu_{x_1,t}^+(s) + b \int_{[0,1]} f(s) d\mu_{x_2,t}^+(s) \quad \mu\text{-a.e.} \end{aligned}$$

It follows that the map $x \rightarrow \int_{[0,1]} \cdot d\mu_{x,\cdot}^+$ is an element of $B(X, B(L_1, L_1([0, 1], X)))$. Consequently, by Theorem 1, there exists an operator $T_+ \in B(L_1([0, 1], X))$ whose representing family of measures is $\{\mu_{x,t}^+\}$. If we now define $P_+(T) = T_+$ for each $T \in B(L_1([0, 1], X))$, then P_+ is a projection on $B(L_1([0, 1], X))$ of norm 1. Now fix u in $[0, 1]$ and T in $B(L_1([0, 1], X))$. Then for each f in L_1 and x in X we have for almost all $t > u$ that

$$(T_+ P_{[0,u]})(fx)(t) = \int_{[t,1]} (P_{[0,u]}f)(s) d\mu_{x,t}(s) = 0.$$

Hence $P_{[0,u]}T_+P_{[0,u]} = T_+P_{[0,u]}$ and the range of P_+ consists of upper triangular operators. Conversely, let T be an upper triangular operator with representing measures $\{\mu_{x,t}\}_{x \in X, t \in [0, 1]}$. Then for each Borel set E , x in X , and u in $[0, 1]$ we have that

$$P_{[0,u]}TP_{[0,u]}(x\chi_E) = TP_{[0,u]}(x\chi_E)$$

and hence

$$\chi_{[0,u]}(t)\mu_{x,t}(E \cap [0, u]) = \mu_{x,t}(E \cap [0, u])$$

for almost all t . Thus, for almost all $t > u$, $\mu_{x,t}(E \cap [0, u]) = 0$. It follows easily that $\mu_{x,t}(E \cap [0, t]) = 0$ for almost all $t \in [0, 1]$. It follows that $P_+(T) = T$ so the range of P_+ consists exactly of the upper triangular operators. If we now let $P_- = I - P_+$, then it is easy to check that P_- is a projection of norm 1 whose range consists of lower triangular operators. To see that P_- is a projection onto the strictly lower triangular operators, let T be in $B(L_1([0, 1], X))$ and put $T_- = P_-(T)$. If $\{\mu_{x,t}\}_{x \in X, t \in [0, 1]}$ is the family of representing measures for T , then it follows that $\{\mu_{x,t}(\cdot \cap [0, t])\}_{x \in X, t \in [0, 1]}$ is the family of representing measures for T_- . Now for each f in L_1 and x in X , we have by Proposition 6.1(iii) of [10] that $(P_dT_-)(fx)(t) = f(t)\mu_{x,t}(\{t\} \cap [0, t]) = 0$ a.e. Hence $P_d(T_-) = 0$ and the proof is complete.

(iv) We first note that the operator $T_d = P_d(T)$ is bounded below. If not, then there exist functions f_n in $L_1([0, 1], X)$ with $\|f_n\| = 1$ for all n such that $\lim_n \|T_d f_n\| = 0$. Since $\|T_d f_n\| \geq \|(T - T_d)f_n\|$ we have that $\lim_n \|(T - T_d)f_n\| = 0$ and hence $\lim_n \|T f_n\| = 0$, which contradicts the invertibility of T . We next show that for λ , $0 < \lambda < 1$, the operator $T_d + \lambda(T - T_d)$ is bounded below. If not, there again exists for each n a function f_n in $L_1([0, 1], X)$ with $\|f_n\| = 1$ such that $\lim_n \|T_d f_n + \lambda(T - T_d)f_n\| = 0$. By passing to a suitable sequence, we may assume that both $\lim_n \|T_d f_n\|$ and $\lim_n \|(T - T_d)f_n\|$ exist and hence $\lim_n \|T_d f_n\| = \lambda \lim_n \|(T - T_d)f_n\|$. But $\lim_n \|T_d f_n\| \geq \lim_n \|(T - T_d)f_n\|$ and neither side of the inequality is zero. Hence $\lambda \geq 1$, a contradiction. Now consider the set $A = \{\lambda \in [0, 1] | T_d + \lambda(T - T_d) \text{ is invertible}\}$.

We have that A is an open nonempty subset of $[0, 1]$. But A is also closed since an operator that is bounded below and the limit of a sequence of invertible operators must itself be invertible [12, Lemma 3.2]. Consequently, $A = [0, 1]$. In particular, $0 \in A$ so T_d is invertible. Now since T_d is diagonal, there exists a strongly Borel measurable function $g: [0, 1] \rightarrow B(X)$ such that $(T_d f)(t) = g(t)f(t)$ a.e. Hence if we take any x in X and any Borel set E ,

$$\frac{1}{\mu E} \int_E \|g(t)x\| d\mu(t) = \int_{[0,1]} \left\| T_d \left(x \frac{\chi_E}{\mu E} \right) (t) \right\| d\mu(t) \geq \|T_d^{-1}\|^{-1} \|x\|.$$

Hence for each x in X there is a null set N_x in $[0, 1]$ such that $\|g(t)x\| \geq \|T_d^{-1}\|^{-1} \|x\|$ for all $t \notin N_x$. Since X is separable, it quickly follows that there is a null set N in $[0, 1]$ such that if $t \notin N$, $\|g(t)x\| \geq \|T_d^{-1}\|^{-1} \|x\|$ for all x in X . Hence, for almost all t , the operators $g(t)$ are uniformly bounded below. Now let x be any element of X . Then since $T_d T_d^{-1} = I$, we have that there is a null set M_x in $[0, 1]$ such that for $t \notin M_x$,

$$x \chi_{[0,1]}(t) = (T_d T_d^{-1} (x \chi_{[0,1]}))(t) = g(t) T_d^{-1} (x \chi_{[0,1]})(t).$$

Again since X is separable, there is a null set M in $[0, 1]$ such that if $t \notin M$, then the operator $g(t)$ is onto. Thus for $t \notin M \cup N$ the operator $g(t)$ is invertible. Hence if we define $h: [0, 1] \rightarrow B(X)$ by

$$h(t) = \begin{cases} 0, & t \in M \cup N, \\ g(t)^{-1}, & \text{otherwise,} \end{cases}$$

then h is a strongly Borel measurable map and $(T_d^{-1} f)(t) = h(t)f(t)$ a.e. so T_d^{-1} is diagonal.

III. The factorization theorem. In this section we prove our main result.

THEOREM 3. *Let T be a diagonally dominant invertible operator on $L_1([0, 1], X)$. Then T has a unique LU-factorization.*

Our method of proof follows the same path as in [12] which in turn was inspired by [7]. We require several preliminary lemmas before we give the proof of Theorem 3. For each operator A on $L_1([0, 1], X)$, we define operators P_A, Q_A on $B(L_1([0, 1], X))$ by $P_A(T) = (AT)_+$ and $Q_A(T) = (TA)_-$ for T in $B(L_1([0, 1], X))$. Our goal is to show that if $\|A\| \leq 1$ and $I - A$ is invertible, then both of the operators $\hat{I} - P_A$ and $\hat{I} - Q_A$ are invertible, where \hat{I} is the identity operator on $B(L_1([0, 1], X))$. The first lemma is needed in both of these efforts.

LEMMA 4. *Let T be in $B(L_1([0, 1], X))$ and let $\varepsilon > 0$. Then for each measurable set $G \subset [0, 1]$ of positive measure and each $x_0 \in X$ with $\|x_0\| = 1$ there exists a countable collection of disjoint measurable subsets I_n of G and numbers b_n in $[0, 1]$ so that $\sum_n \mu I_n = \mu G$ and for each n , $\mu I_n > 0$ and $\|(T_+ - P_{[0, b_n]})(x_0 \chi_{I_n} / \mu I_n)\| < \varepsilon$.*

PROOF. Let $\varepsilon > 0$ and let G and x_0 be given as above. It suffices to show that there is a measurable set $E \subset G$ and a number d in $[0, 1]$ with $\mu E > 0$ and $\|(T_+ - P_{[0, d]}T)(x_0 \chi_E / \mu E)\| < \varepsilon$; for once this is done we may appeal to the technique of exhaustion to produce the I_n 's which satisfy the conclusion. Now let $\{\mu_{x,t}\}_{x \in X, t \in [0, 1]}$ be the family of representing measures for T . By Theorem 1, the function $t \rightarrow |\mu_{x_0, t}|[0, 1]$ is integrable. Hence there exists a $\delta > 0$ with $\delta < \varepsilon$ such that if B is a measurable set with $\mu B < \delta$, then

$$\int_B |\mu_{x_0, t}|[0, 1] d\mu(t) < \frac{\varepsilon}{2} \mu G.$$

Now, for each n , consider the function $f_n: [0, 1] \rightarrow R$ given by

$$f_n(t) = |\mu_{x_0, t}| \left(\left[t - \frac{1}{n}, t \right) \cap [0, 1] \right).$$

Note that each f_n is measurable (since, for example, it is the pointwise limit of the measurable functions

$$f_{n,m}(t) = \sum_{j=0}^{2^m} |\mu_{x_0, t}| \left(\left[\frac{j}{2^m} - \frac{1}{n}, \frac{j}{2^m} \right) \cap [0, 1] \right) \chi_{[j/2^m, (j+1)/2^m)}(t)).$$

Moreover, $\lim_n f_n(t) = 0$ for each t in $[0, 1]$. Choose α such that $2\alpha/(1 - \alpha) < \varepsilon/2$ and $0 < \alpha < 1$. By Ergorov's theorem there exists an integer N such that $\mu\{t: |f_N(t)| > \alpha\} < \delta$. By the regularity of μ there exists an open set $O_1 \supset G$ such that $\mu G > (1 - \alpha)\mu O_1$. Choose disjoint open intervals A_i such that $\bigcup_{i=1}^{+\infty} A_i \subset O_1$, $\sum_i \mu A_i = \mu O_1$ and for each i , $\mu A_i < 1/N$. Let $J = \{i: \mu(A_i \cap G) > ((1 - \alpha)/2)\mu A_i\}$. Then

$$\sum_{i \in J} \mu A_i \geq \left(\frac{1 - \alpha}{2} \right) \mu O_1,$$

since otherwise

$$\begin{aligned} \mu G &= \sum_i \mu(A_i \cap G) = \sum_{i \in J} \mu(A_i \cap G) + \sum_{i \notin J} \mu(A_i \cap G) \\ &< \left(\frac{1 - \alpha}{2} \right) \mu O_1 + \left(\frac{1 - \alpha}{2} \right) \sum_{i \notin J} \mu A_i \\ &< \left(\frac{1 - \alpha}{2} \right) \mu O_1 + \left(\frac{1 - \alpha}{2} \right) \mu O_1 = (1 - \alpha)\mu O_1, \end{aligned}$$

which is a contradiction. Note that $\mu(G \cap (\bigcup_{i \notin J} A_i)) < ((1 - \alpha)/2)(1 - \alpha)^{-1} \mu G$. Now repeat this procedure, i.e. choose an open set O_2 such that $G \cap (\bigcup_{i \notin J} A_i) \subset O_2 \subset \bigcup_{i \notin J} A_i$ and $\mu(G \cap (\bigcup_{i \notin J} A_i)) > (1 - \alpha)\mu O_2$ and disjoint open intervals C_k such that $\bigcup_{k=1}^{+\infty} C_k \subset O_2$, $\sum_k \mu C_k = \mu O_2$ and for each k , $\mu C_k < 1/N$. If we let $L = \{k: \mu(C_k \cap G) > ((1 - \alpha)/2)\mu C_k\}$, then

$$\sum_{k \in L} \mu C_k \geq \left(\frac{1 - \alpha}{2} \right) \mu O_2.$$

Note that

$$\mu \left(G \cap \left(\bigcup_{k \notin L} C_k \right) \right) < \left(\frac{1-\alpha}{2} \right)^2 (1-\alpha)^{-1} \mu G.$$

Continuing in this way we can obtain a sequence of disjoint open intervals F_m so that, for each m , $\mu(F_m \cap G) > ((1-\alpha)/2)\mu F_m$ and $\mu F_m < 1/N$. Moreover, $G \subset \bigcup F_m$ a.e., i.e. $\mu(G \setminus \bigcup_m F_m) = 0$. Now for each m let $B_m = F_m \cap \{t: |f_N(t)| > \alpha\}$ and $B = \bigcup_m B_m$. Then $\mu B < \delta$ and hence, for some m ,

$$\int_{B_m} |\mu_{x_0,t}|[0,1] d\mu(t) < \frac{\varepsilon}{2} \mu(F_m \cap G).$$

If not, then

$$\begin{aligned} \int_B |\mu_{x_0,t}|[0,1] d\mu(t) &= \sum \int_{B_m} |\mu_{x_0,t}|[0,1] d\mu(t) \\ &\geq \sum_m \frac{\varepsilon}{2} \mu(F_m \cap G) = \frac{\varepsilon}{2} \mu G, \end{aligned}$$

which contradicts the choice of δ . Fix an m such that

$$\int_{B_m} |\mu_{x_0,t}|[0,1] d\mu(t) < \frac{\varepsilon}{2} \mu(F_m \cap G).$$

Now $F_m = (c,d)$ for some numbers c,d . If we let $E_m = F_m \cap G$, then

$$\begin{aligned} (T_+ - P_{[0,d]}T) \left(\frac{x_0 \chi_{E_m}}{\mu E_m} \right) (t) &= \frac{1}{\mu E_m} \mu_{x_0,t}(E_m \cap [t,1]) \\ &\quad + \frac{-1}{\mu E_m} \mu_{x_0,t}(E_m) \chi_{[0,d]}(t) \quad \text{a.e.} \\ &= \frac{-1}{\mu E_m} \mu_{x_0,t}(E_m \cap [0,t)) \chi_{[0,d]}(t) \quad \text{a.e.} \\ &= \frac{-1}{\mu E_m} \mu_{x_0,t}(E_m \cap [0,t)) \chi_{[c,d]}(t) \quad \text{a.e.} \end{aligned}$$

Thus

$$\begin{aligned} \left\| (T_+ - P_{[0,d]}T) \left(\frac{x_0 \chi_E}{\mu E_m} \right) \right\| &= \frac{1}{\mu E_m} \int_{F_m} \|\mu_{x_0,t}(E_m \cap [0,t))\| d\mu(t) \\ &\leq \frac{1}{\mu E_m} \int_{F_m} |\mu_{x_0,t}|(c,t) d\mu(t) \\ &\leq \frac{1}{\mu E_m} \int_{F_m} |\mu_{x_0,t}|[t - \frac{1}{N}, t) d\mu(t) \\ &\leq \frac{1}{\mu E_m} \left[\int_{F_m \setminus B_m} |\mu_{x_0,t}|[t - \frac{1}{N}, t) d\mu(t) \right. \\ &\quad \left. + \int_{B_m} |\mu_{x_0,t}|[t - \frac{1}{N}, t) d\mu(t) \right] \\ &\leq \frac{1}{\mu E_m} \alpha \mu(F_m \setminus B_m) + \frac{\varepsilon}{2} \leq \frac{2\alpha \mu(F_m \setminus B_m)}{(1-\alpha)\mu F_m} + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} \frac{\mu(F_m \setminus B_m)}{\mu F_m} + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

Now taking $E = E_m$ yields the desired conclusion.

LEMMA 5. Let A be in $B(L_1([0, 1], X))$. If $\|A\| \leq 1$ and $I - A$ is an invertible operator, then $\hat{I} - P_A$ is an invertible operator.

PROOF. We note first that $\hat{I} - P_A$ is the limit of the invertible operators $\hat{I} - P_{\lambda A}$ as $\lambda \uparrow 1$. Hence by Lemma 3.2 of [12] if $\hat{I} - P_A$ is bounded below, then $\hat{I} - P_A$ is invertible. Suppose then that $\hat{I} - P_A$ is not bounded below. Then, for each $\varepsilon > 0$, there is a $T \in B(L_1[0, 1], X)$ with $\|T\| = 1$ such that $\|(\hat{I} - P_A)(T)\| < \varepsilon$. Now $\|(\hat{I} - P_A)(T)\| = \sup_{\|f\| \leq 1} \|(T - (AT)_+)(f)\|$. Since step functions are dense in $L_1([0, 1], X)$, there exist open intervals G_i in $[0, 1]$, elements x_i in X , and scalars α_i such that $\|x_i\| = 1$, $\sum_{i=1}^n |\alpha_i| = 1$ and

$$\left\| AT \left(\sum_{i=1}^n \alpha_i x_i \frac{\chi_{G_i}}{\mu_{G_i}} \right) \right\| > (1 - \varepsilon) \|AT\|.$$

It follows that for at least one i , $\|AT(x_i \chi_{G_i} / \mu_{G_i})\| > (1 - \varepsilon) \|AT\|$. Fix such an i .

An appeal to Lemma 4 now produces a countable collection of disjoint measurable subsets I_n of G_i with positive measure and numbers b_n in $[0, 1]$ so that

$$\sum_n \mu(I_n) = \mu(G_i) \quad \text{and} \quad \left\| ((AT)_+ - P_{[0, b_n]} AT) \frac{x_i \chi_{G_i}}{\mu(G_i)} \right\| < \varepsilon$$

for all n . Since

$$x_i \frac{\chi_{G_i}}{\mu(G_i)} = \sum_n \frac{\mu(I_n)}{\mu(G_i)} \frac{x_i \chi_{I_n}}{\mu(I_n)} \quad \text{a.e.,}$$

it follows that for at least one n that $\|AT(x_i \chi_{I_n} / \mu(I_n))\| > (1 - \varepsilon) \|AT\|$. Fix such an n and let $f = x_i \chi_{I_n} / \mu(I_n)$ and $b = b_n$. Then we have $\|((AT)_+ - P_{[0, b]} AT)(f)\| < \varepsilon$ and $\|ATf\| > (1 - \varepsilon) \|AT\| \geq (1 - \varepsilon)^2$. Note that $\|Tf\| > (1 - \varepsilon)^2$ since $\|A\| \leq 1$ and note that $\|(T - P_{[0, b]} AT)(f)\| < 2\varepsilon$. Since $\|g\| = \|P_{[0, b]} g\| + \|P_{(b, 1]} g\|$ for any g in $L_1([0, 1], X)$, we have that

$$\begin{aligned} \|P_{(b, 1]} ATf\| &= \|ATf\| = \|P_{[0, b]} ATf\| \\ &\leq \|Tf\| = \|P_{[0, b]} ATf\| \leq \|Tf - P_{[0, b]} ATf\| < 2\varepsilon. \end{aligned}$$

But then

$$\begin{aligned} \|(I - A)Tf\| &= \|Tf - P_{[0, b]} ATf - P_{(b, 1]} ATf\| \\ &\leq \|Tf - P_{[0, b]} ATf\| + \|P_{(b, 1]} ATf\| \leq 4\varepsilon \end{aligned}$$

and yet $\|Tf\| > (1 - \varepsilon)^2$. This contradicts the invertibility of $I - A$ and completes the proof.

To show that $\hat{I} - Q_A$ is invertible requires the following preliminary result.

LEMMA 6. Let $T, A \in B(L_1([0, 1], X))$ with $\|T\| = \|A\| = 1$. If there is an f in $L_1([0, 1], X)$ with $\|f\| \leq 1$ and a $\delta > 0$ such that $\|Tf\| > 1 - \delta$ and $\|T - (TA)_-\| < \delta$, then $\|Tf - T Af\| < 5\sqrt{\delta}$.

PROOF. The conclusion is obvious if $\delta \geq 1$ so we may suppose that $0 < \delta < 1$. We may also assume f is a step function. By repeated applications of Lemma 4, we may write f in the form

$$\sum_{i=1}^{+\infty} \alpha_i x_i \frac{\chi_{E_i}}{\mu_{E_i}},$$

where the E_i 's are a countable collection of disjoint measurable sets having positive measure such that

$$\mu\left(\bigcup_{i=1}^{+\infty} E_i\right) = 1, \quad \|x_i\| = 1, \quad \sum_{i=1}^{+\infty} |\alpha_i| \leq 1,$$

and

$$\left\| ((TA)_+ - P_{[0, b_i]} TA) \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| < \sqrt{\delta}$$

for some numbers b_i in $[0, 1]$. Now let $J = \{i: \|T(x_i \chi_{E_i} / \mu E_i)\| > 1 - \sqrt{\delta}\}$. Since $\|T - (TA)_-\| < \delta$, we have, for all $i \in J$, that

$$\left\| (TA)_- \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| > 1 - \sqrt{\delta} - \delta \geq 1 - 2\sqrt{\delta}.$$

Hence

$$\begin{aligned} 1 - 2\sqrt{\delta} &\leq \left\| (TA)_- \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| = \left\| ((TA) - (TA)_+) \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| \\ &\leq \left\| (TA - P_{[0, b_i]} TA) \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| + \sqrt{\delta}. \end{aligned}$$

Since $\|(TA)(x_i \chi_{E_i} / \mu E_i)\| \leq 1$ it follows that $\|P_{[0, b_i]} TA(x_i \chi_{E_i} / \mu E_i)\| \leq 3\sqrt{\delta}$ and so $\|(TA)_+(x_i \chi_{E_i} / \mu E_i)\| < 4\sqrt{\delta}$. Next note that

$$\begin{aligned} 1 - \delta < \|Tf\| &\leq \left\| \sum_{i \in J} \alpha_i T \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| + \left\| \sum_{i \notin J} \alpha_i T \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| \\ &\leq \left(1 - \sum_{i \notin J} |\alpha_i| \right) + (1 - \sqrt{\delta}) \left(\sum_{i \notin J} |\alpha_i| \right) \\ &\leq 1 - \sqrt{\delta} \sum_{i \notin J} |\alpha_i|. \end{aligned}$$

This implies that $\sum_{i \notin J} |\alpha_i| < \sqrt{\delta}$ and consequently

$$\begin{aligned} \|(TA)_+(f)\| &= \left\| \sum_i \alpha_i (TA)_+ \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| \\ &\leq \sum_{i \in J} |\alpha_i| \left\| (TA)_+ \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| + \sum_{i \notin J} |\alpha_i| \left\| (TA)_+ \left(\frac{x_i \chi_{E_i}}{\mu E_i} \right) \right\| \\ &\leq 3\sqrt{\delta} \left(\sum_{i \in J} |\alpha_i| \right) + \sum_{i \notin J} |\alpha_i| \leq 4\sqrt{\delta}. \end{aligned}$$

The desired result now follows easily since

$$\|Tf - T Af\| \leq \|(T - (TA)_-)(f) - (TA)_+(f)\| \leq \sqrt{\delta} + 4\sqrt{\delta} = 5\sqrt{\delta}.$$

LEMMA 7. *Let A be in $B(L_1([0, 1], X))$. If $\|A\| \leq 1$ and $I - A$ is an invertible operator, then $\hat{I} - Q_A$ is an invertible operator.*

PROOF. Just as in Lemma 5, it suffices to show that $\hat{I} - Q_A$ is bounded below. So we suppose by way of contradiction that $\hat{I} - Q_A$ is not bounded below. Then

for each $\varepsilon > 0$ there exists a T in $B(L_1([0, 1], X))$ with $\|T\| = 1$ such that

$$\|T - (TA)_-\| = \|(\hat{I} - Q_A)(T)\| < \varepsilon.$$

Choose a g in $L_1([0, 1], X)$ with $\|g\| = 1$ such that $\|Tg\| > 1 - \varepsilon$. An appeal to Lemma 6 with $f = g$ and $\delta = \varepsilon$ insures that $\|Tg - TAg\| < 5\sqrt{\varepsilon}$. It follows that $\|TAg\| > 1 - \varepsilon - 5\sqrt{\varepsilon} > 1 - 25\sqrt{\varepsilon}$. Since $\|Ag\| \leq 1$, we may apply Lemma 6 again with $f = Ag$ and $\delta = 25\sqrt{\varepsilon}$ to conclude that

$$\|TAg - TA^2g\| < 5(5\varepsilon^{1/4})$$

and hence $\|TA^2g\| > 1 - 25\varepsilon^{1/2} - 25\varepsilon^{1/4} \geq 1 - 5^4\varepsilon^{1/4}$ provided $25\sqrt{\varepsilon} < 1$. In general, one obtains that for each positive integer n

$$\|TA^n g\| > 1 - 5^{2n}\varepsilon^{(1/2)^n}$$

and

$$\|TA^n g - TA^{n+1}g\| < 5^{n+1}\varepsilon^{(1/2)^{n+1}}$$

provided $5^{2n}\varepsilon^{(1/2)^n} < 1$. Now for an integer m such that $5^{2m}\varepsilon^{(1/2)^m} < 1$ we define $u_m = g + Ag + \cdots + A^m g / m + 1$ and

$$f(\varepsilon, m) = \varepsilon + \frac{1}{m+1} \sum_{i=1}^m \sum_{j=0}^{i-1} 5^{j+1} \varepsilon^{(1/2)^{j+1}}.$$

Then

$$\begin{aligned} \|u_m\| &\geq \|Tu_m\| = \frac{1}{m+1} \|Tg + TAg + \cdots + TA^m g\| \\ &\geq \|Tg\| - \frac{1}{m+1} \sum_{i=1}^m \|TA^i g - Tg\| \\ &\geq 1 - \varepsilon - \frac{1}{m+1} \sum_{i=1}^m \sum_{j=0}^{i-1} \|TA^{j+1} g - TA^j g\| \\ &> 1 - \varepsilon - \frac{1}{m+1} \sum_{i=1}^m \sum_{j=0}^{i-1} 5^{j+1} \varepsilon^{(1/2)^{j+1}} = 1 - f(\varepsilon, m). \end{aligned}$$

But $(I - A)u_m = g - A^{m+1}g / (m + 1)$ and hence

$$\|u_m\| \leq \frac{1}{m+1} \|(I - A)^{-1}\| \|g - A^{m+1}g\| \leq \frac{2}{m+1} \|(I - A)^{-1}\|.$$

It follows that $1 - f(\varepsilon, m) \leq (2/(m+1))\|(I - A)^{-1}\|$. But for fixed m , $\lim_{\varepsilon \rightarrow 0} f(\varepsilon, m) = 0$. Hence the above inequality must fail for m large and ε close to zero. This contradiction completes the proof.

We may now give the proof of the main result.

PROOF OF THEOREM 3. Let T be a diagonally dominant operator. By Proposition 2, the operator T_d is invertible. Now let $S = TT_d^{-1}$. Then S is an invertible operator and, again by Proposition 2, $S_d = P_d(TT_d^{-1}) = (P_d T)T_d^{-1} = I$. Moreover, S is a diagonal dominant operator since if f is an element of $L_1([0, 1], X)$ we have that

$$\|(S - S_d)(f)\| = \|(TT_d^{-1} - T_d T_d^{-1})(f)\| = \|(T - T_d)(T_d^{-1} f)\| \leq \|TT_d^{-1}\| = \|S_d f\|.$$

Consequently, S can be written in the form $I - A$, where $\|A\| \leq 1$. It suffices to show that S has an LU -factorization. Toward this end, note that by Lemmas 5 and 7 the operators $\hat{I} - P_A$ and $\hat{I} - Q_A$ are invertible. Hence there exist operators V, W in $B(L_1[0, 1], X)$ such that $(\hat{I} - P_A)(V) = I$ and $(\hat{I} - Q_A)(W) = I$. We claim that the operators V and W are (bilaterally) invertible. To see this, note first that $V = I + (AV)_+$ and $W = I + (WA)_-$. From this it follows that $(I - A)V = I - (AV)_-$ and $W(I - A) = I - (WA)_+$. Consequently,

$$(I - (WA)_+)(I + (AV)_+) = W(I - A)(V) = (I + (WA)_-)(I - (AV)_-)$$

or

$$(AV)_+ - (WA)_+(AV)_+ - (WA)_+ = (WA)_- - (WA)_-(AV)_- - (AV)_-.$$

But the operator on the left is upper triangular while that on the right is strictly lower triangular. Hence both sides are zero. It follows that $W(I - A)V = I$ and so W is left invertible and V is right invertible. Now consider the curves of operators $\{(\hat{I} - P_{\lambda A})^{-1}(I) : 0 \leq \lambda \leq 1\}$ and $\{(\hat{I} - Q_{\lambda A})^{-1}(I) : 0 \leq \lambda \leq 1\}$ which initiate at $I(\lambda = 0)$ and terminate at V and W respectively ($\lambda = 1$). Since I is (bilaterally) invertible and each element of the two curves is unilaterally invertible we have by [2] that all elements are (bilaterally) invertible. Hence W and V are (bilaterally) invertible. Hence we have the factorization $I - A = W^{-1}V^{-1}$, where $W^{-1} = (I - A)V = I - (AV)_-$ and $W = I + (WA)_-$ are lower triangular operators $V^{-1} = W(I - A) = I - (WA)_+$ and $V = I + (AV)_+$ are upper triangular operators. Finally, note that $(W^{-1})_d = W_d = I$ and so if we let $L = W^{-1}$ and $U = V^{-1}$ we have the desired factorization. If $L'U'$ is another LU -factorization of T then it follows that $L^{-1}L' = U(U')^{-1}$. Since the left side is lower triangular while the right side is upper triangular, it follows that $L^{-1}L' = U(U')^{-1} = D$ for some diagonal operator D . But then $L' = LD$ and hence by Proposition 1

$$I = P_d(L') = P_d(LD) = P_d(L)(D) = ID = D$$

so $L' = L$ and $U' = U$. This completes the proof.

We remark that for any operator T in $B(L_1([0, 1], X))$ the operator $T + \lambda I$ is diagonally dominant and invertible for sufficiently large λ . Hence Theorem 3 asserts that any operator on $L_1([0, 1], X)$ has scalar translates which have an LU -factorization.

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